3901. A moving particle is modelled, for time $t \in [0, \infty)$, by the parametric equations

$$x = (t-3)^2,$$
$$y = (t-2)^3 - (t-2).$$

(a) Show that, when the particle is at the origin, it is moving in the y direction.

(b) Find the point at which the particle crosses the x axis for the first time, and show that this first crossing takes place at angle $\theta = \arctan \frac{1}{6}$.

3902. In this question, you should assume that $\sqrt{n}$ is irrational for any $n \in \mathbb{N}$ which is not a perfect square.

Prove that $\sqrt{2} - \sqrt{3}$ is irrational.

3903. Let a and b be constant integers between 1 and 100 inclusive, and let sets A and B be

$$A = \{n \in \mathbb{Z} : n > a\},$$
$$B = \{n \in \mathbb{Z} : n < b\}.$$

You are given that $A \cap B$ has 20 elements.

(a) Find the value of $b - a$.

(b) Find the set of possible values of $a + b$.

3904. Below is a generalised binomial expansion, which is assumed to converge:

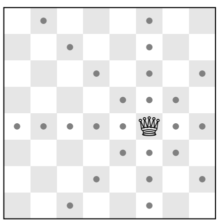
$$\frac{p}{(4+qx^2)^3} \equiv 1 + \tfrac{3}{4}x^2 + rx^4 + ...$$

Determine the value of the constants p, q, r.

3905. (a) Show that $S = n^3 + (n+1)^3 + (n+2)^3$ may be written as $S = 3(n+1)(n^2+2n+3)$.

(b) Prove that the sum of three consecutive cubes greater than eight has at least eight distinct factors.

3906. In chess, queens threaten squares as shown.



Two queens are placed at random on distinct squares of a chessboard. Find the probability that they threaten each other.

3907. Show that the tangent to the quartic $y = x^4 - x$ at $x = 1$ does not intersect the curve again.

3908. Prove that there is an irrational number between any two distinct rational numbers.

3909. A curve is given by the relation

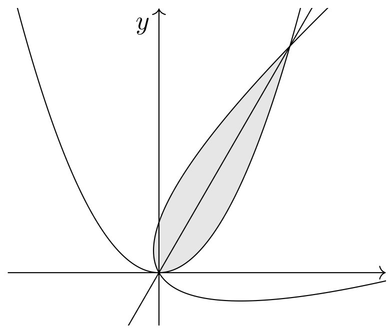
$$\sqrt{x+y} + \sqrt{x-y} = 1.$$

(a) By differentiating implicitly, show that

$$\left(1 + \tfrac{dy}{dx}\right)\sqrt{x-y} + \left(1 - \tfrac{dy}{dx}\right)\sqrt{x+y} = 0.$$

(b) Hence, determine the coordinates of the point at which the gradient is 1.

3910. Curves C_1 and C_2 are defined as $y = x^2$ and its image under rotation by 60° clockwise around O.



(a) Show that the intersections of C_1 and C_2 lie on the line $y = \sqrt{3}x$.

(b) Hence, show that the region enclosed by C_1 and C_2 has area $\sqrt{3}$.

3911. The classical *momentum* of an object is defined as the vector quantity mv, where v is the velocity.

In 1D, two particles of mass m_1 and m_2, travelling at velocities u_1 and u_2, interact. For a duration t, they exert a constant force F on each other. They end up with velocities v_1 and v_2.

Show, using Newton's laws and *suvat*, that total momentum is conserved, i.e. that total momentum before interaction is the same as total momentum after interaction.

3912. A differential equation is given as

$$\frac{dy}{dx} = 2xy.$$

Prove that, if $y = \mathrm{f}(x)e^{x^2}$ satisfies the differential equation, then the function f must be constant.

3913. Find the limit of $\dfrac{600x^2 - 1470x + 441}{50x^2 - 135x + 63}$

(a) as $x \to \infty$,

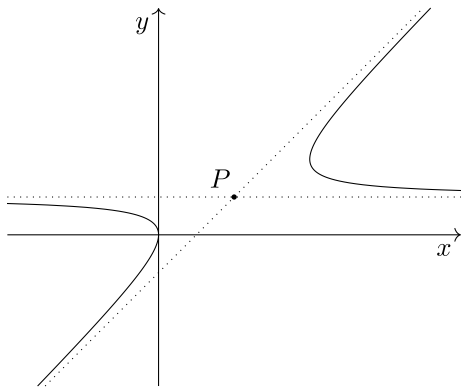
(b) as $x \to \frac{21}{10}$.

3914. Determine the set of x values for which the graph $y = x^4 - x^2$ is both increasing and concave.

3915. Show that, for small x,

$$\frac{1}{1-\sin x+2\sin^2 x}\approx 1+x-x^2.$$

3916. The graph $xy-x-y^2=0$ is shown, with two asymptotes marked as dotted lines. The equation of the oblique asymptote is $y=x-1$.



(a) Determine algebraically the coordinates of the two points on the curve at which the gradient is undefined.

(b) Show that the horizontal asymptote is $y=1$, and write down the coordinates of P.

(c) Find the set of values of m for which the line $y-1=m(x-2)$ intersects the curve.

3917. Show carefully that $\tan x$ is never stationary with respect to $\sin x$.

3918. Find the probability that, if two distinct roots of the equation $6x^4-13x^3-2x^2+7x+2=0$ are chosen at random, exactly one of them is positive.

3919. The *complex unit* i is defined as having $i^2=-1$. Verify that the quadratic equation $z^2-8z+20=0$ has no real roots, but is satisfied by $z=4\pm 2i$.

3920. A curve is given as

$$y=\frac{e^x-1}{e^x+1}.$$

(a) Determine the behaviour as $x\to\pm\infty$.

(b) Show that the curve has no stationary points.

(c) Show that, if (a,b) is a point on the curve, then $(-a,-b)$ is also a point on the curve.

(d) Hence, sketch the curve.

3921. Show that $\displaystyle\int_0^{\frac{1}{6}\pi}\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\,dx=\tfrac{2\sqrt{3}}{3}$.

3922. Solve the following equation exactly:

$$3(\ln x)^2-13\ln x+16=\frac{4}{\ln x}.$$

3923. A function f has domain $\mathbb{R}$ and range $[0,1]$. State, with a reason, whether each of the following is necessarily true:

(a) $\text{f}^2(x)\in[0,1]$ for all $x\in\mathbb{R}$,

(b) f^2 has range $[0,1]$ over $\mathbb{R}$.

3924. A relation is given, for constant k, as

$$\frac{x}{x+y}+\frac{x+y}{y}=k.$$

(a) Show that $x^2+(3-k)xy+(1-k)y^2=0$.

(b) Hence, show carefully that, whatever the value of k, the locus of the relation is a distinct pair of straight lines through the origin.

3925. In a card game, player A takes one of player B's cards, at random, and, simultaneously, vice versa. Before the swap, A has two Jacks and Queen, and B has two Queens and a King.

(a) Draw a table to show the nine possible swaps.

(b) Given that B ends up with at least a pair, find the probability that A does too.

3926. A biquadratic is a curve of the form

$$y=ax^4+bx^2+c.$$

Sketch biquadratics with the following coefficients:

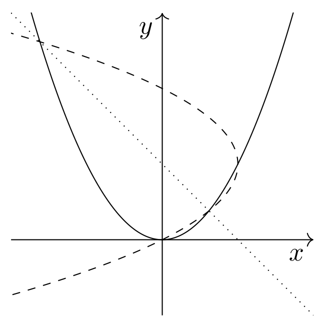
(a) $a=1,b=-1,c=0$,

(b) $a=1,b=-2,c=1$.

3927. Eliminate t from the following equations to find a simplified relationship between x and y:

$$x=\operatorname{cosec}2t,$$
$$y=\tan 2t.$$

3928. The curve $y=x^2$ is reflected in the line $x+y=k$.



Give the equation of the transformed curve.

3929. A triangle has angles $\arccos\frac{11}{14}$, $\arccos\frac{13}{14}$ and θ.

(a) Show that $\sin(\arccos x)\equiv\sqrt{1-x^2}$.

(b) Using the identity $\sin(\pi-x)\equiv\sin x$ and a compound-angle formula, show algebraically that $\sin\theta={\sqrt{3}}/{2}$.

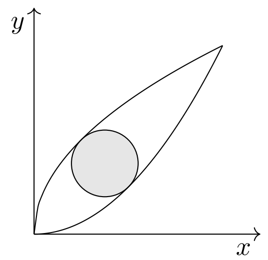
3930. Show that the sum of the integers from 1 to $3k$ which are not divisible by 3 is $3k^2$.

3931. In an electrical circuit, n switches are each either off or on. The probability of each being on is $1/n$, independently of the others.

(a) Write down the distribution of X, the number of switches that are on.

(b) The states of the switches are now observed. It is seen that X is no greater than $\mathbb{E}(X)$. Find the probability that X is equal to $\mathbb{E}(X)$, giving your answer in simplified form.

3932. The graphs $y = x^2$ and $y = \sqrt{x}$ are drawn below, on the domain $[0, 1]$, with a shaded circle that is tangent to both.



Prove that area of the circle satisfies $A \leq \frac{\pi}{32}$.

3933. You are given that the following equations hold:

$$p + \sqrt{4 + pq} + q = 0,$$
$$p + 4q = 6.$$

Determine the value of p and the value of q. There is only one (p, q) pair which satisfies the equations.

3934. Sketch the graphs

(a) $y = \sin x$,

(b) $y = |\sin x|$,

(c) $|y| = \sin x$,

(d) $|y| = |\sin x|$.

3935. The functions $\text{f}_1(x)$ and $\text{f}_2(x)$ are both solutions of the differential equation $\text{f}(x) = \text{f}'(x)$.

(a) Show that $\dfrac{d}{dx}\left(\dfrac{\text{f}_1(x)}{\text{f}_2(x)}\right) = 0$.

(b) Hence, prove that $\text{f}_1(x) \propto \text{f}_2(x)$.

3936. A function f is defined over the reals, and has range $[a, b]$, where $0 < a < b$. Give the range of each of the following functions:

(a) $x \longmapsto \dfrac{1}{\text{f}(x)}$,

(b) $x \longmapsto \dfrac{\text{f}(x)}{\text{f}(x) + 1}$.

3937. The curve $x = y^2$ has a normal drawn to it at $x = k$, where $k > 0$. Show that this normal crosses the x axis at $x = k + \frac{1}{2}$.

3938. During its take-off, a rocket accelerates at $4g$ ms^{-2} vertically. There are two astronauts in the cockpit, sitting 1 metre apart.

(a) Write down the acceleration, relative to the cockpit, with which projectiles inside it fall.

(b) Relative to the astronauts, find the minimum speed at which a projectile must be thrown to travel 1 metre horizontally.

(c) Compare this value with the equivalent value on Earth.

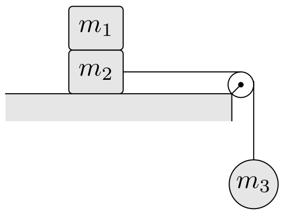
3939. Sketch the curve $y = a^x - a^{2x}$, where $a \in (1, \infty)$, giving the coordinates of all axis intercepts and stationary points.

3940. The *parallelogram law* says that, in a parallelogram $ABCD$, the edges and diagonals satisfy

$$|AC|^2 + |BD|^2 = 2|AB|^2 + 2|BC|^2.$$

Using the cosine rule, prove this result.

3941. Two blocks are stacked on a table. The lower block is connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley, to a sphere which hangs freely. Both contacts with the lower block are rough, with coefficients of friction μ_1 at the upper surface and μ_2 at the lower.



The stack accelerates as a single object. Show that

$$m_3 \leq \frac{(\mu_1 + \mu_2)(m_1 + m_2)}{1 - \mu_1}.$$

3942. State, with a reason, which if any of the symbols $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function g:

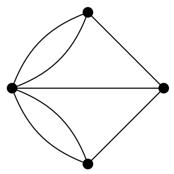
① $\text{g}(x)$ has a factor of $(ax + b)^2$,

② $\text{g}'(x)$ has a factor of $(ax + b)$.

3943. Find $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{x^4 + h} - \sqrt[4]{x^4 - h}}{2h}$.

3944. Three couples sit down at random around a round table. Show that the probability that exactly two couples are sitting together is $\frac{1}{5}$.

3945. In the *Bridges of Königsberg* problem, a network of seven bridges links four land masses. The scenario is represented in schematic fashion in the diagram below. The (impossible!) task is to walk over every bridge exactly once.



(a) The *degree* of each land mass is defined as the number of bridges connecting it. Verify that degree of each of the four land masses is odd.

(b) Hence, prove that a walker cannot walk a closed loop, crossing every bridge exactly once.

3946. Consider the graph $y = \dfrac{ax+b}{cx+d}$, for $a, b, c, d > 0$.

(a) Show that $y = \dfrac{a}{c} + \dfrac{bc - ad}{c(cx+d)}$.

(b) Assuming that $bc - ad > 0$,

i. Find the equations of the asymptotes.

ii. Sketch the curve.

3947. Sketch a counterexample to the following claim: "If a function is convex throughout its domain, then it can have at most two roots."

3948. This question is about using Newton-Raphson to find the reciprocal of a number $p > 0$.

The equation satisfied by the reciprocal of p is

$$\frac{1}{x} - p = 0.$$

(a) Set up the Newton-Raphson iteration for this equation.

(b) Verify that, with $p = 3$ and $x_0 = 0.5$, the N-R iteration converges to the reciprocal.

(c) The tangent line to $y = \frac{1}{x} - p$ at a generic point $x = x_0$ has equation

$$y = -\frac{x}{x_0^2} + \frac{2}{x_0} - p.$$

Show that, if $x_0 \geq \frac{2}{p}$, then the iteration will fail to converge to the reciprocal.

3949. Three simultaneous equations are given as follows:

$$\begin{aligned} x + y + z &= 2 \\ 6x + 4y &= 3 \\ 2x + y - z &= 5. \end{aligned}$$

Show that this set of equations has no solutions.

3950. "An x intercept of $y = \mathrm{f}(x)$ must be an x intercept of $y^2 = \mathrm{f}(x)$." True or false?

3951. A curve C has implicit definition $x^3y + y^2 = 1$, for $y \in (0, \infty)$. You are given that C has

- a stationary point of inflection at $(0, 1)$,
- a non-stationary point of inflection at $x \approx 1$,
- no other SPs or points of inflection.

(a) Show that C may be expressed as

$$2y = -x^3 + \sqrt{x^6 + 4}.$$

(b) Describe the behaviour as $x \to \infty$.

(c) Explain how you know that C consists of two convex sections and one concave section.

(d) Sketch C.

3952. You are given the following set of facts:

$$\begin{aligned} &\mathbb{P}(X) = \mathbb{P}(Y) = \mathbb{P}(Z) = \tfrac{1}{2}, \\ &\mathbb{P}(X \cap Y) = \tfrac{1}{5}, \\ &\mathbb{P}(X \cap Z) = \tfrac{3}{20}, \\ &\mathbb{P}(Y \cap Z) = \tfrac{1}{4}, \\ &\mathbb{P}(X' \cap Y' \cap Z') = \tfrac{1}{20}. \end{aligned}$$

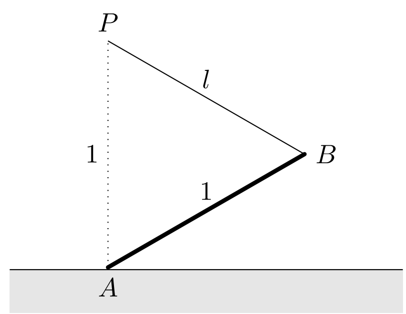
Find $\mathbb{P}(X \cap Y \cap Z)$.

3953. A curve is defined, for $x \in [0, 2\pi)$, by

$$y = \sin^3 x + \cos^3 x.$$

Find the exact coordinates of all stationary points.

3954. A uniform rod AB of mass m and length 1 m is in equilibrium. Point A rests on rough horizontal ground, and point B is attached, by a light string of length l, to a point P, which is 1 m vertically above A. The tension in the string is $T = \frac{1}{2}mgl$.



Show that $\mu \geq \dfrac{l}{\sqrt{4 - l^2}}$.

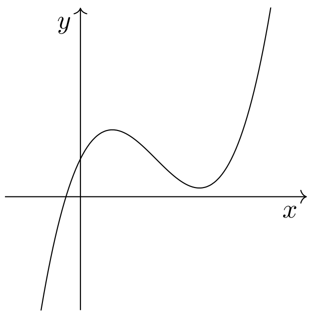
3955. When the quadratic $y = x^2 + 2x - 5$ is reflected in the line $y = -x$, a new parabola is produced. Determine the equation of the transformed graph.

3956. In this question, do not use a calculator.

Determine the solution of $x^4 + x^3 = 2x$.

3957. Using "log" for "ln", as many authors do, answer the mathematical joke: "What's the integral of $\frac{1}{cabin}$ with respect to *cabin*?"

3958. The diagram shows a cubic $y = ax^3 + bx^2 + cx + d$, where $b^2 - 3ac > 0$.



Show **algebraically** that the following procedures give the same x value:

① finding the mean of the stationary points,
② finding the point of inflection.

3959. At time $t = 0$, two boats report their positions and courses, in nautical miles, to a coastguard as:

$$\mathbf{r}_a = 5t\mathbf{i} + (80 - 2t)\mathbf{j}$$
$$\mathbf{r}_b = (120 - 3t)\mathbf{i} + (110 - 4t)\mathbf{j}.$$

(a) Show that, unless there is a change of course, the boats will collide.

(b) The coastguard warns them when they are halfway to collision. Find, to 3sf, the distance between the boats at this time.

3960. The lengths of triangle ABC are varying in time. At an instant when $a = 1$, $b = \sqrt{3}$, $c = 2$, the rates of change of length are $\dot{a} = 0$, $\dot{b} = 2$, $\dot{c} = 0$.

(a) Using the cosine rule, show that

$$\sin B \frac{dB}{dt} = \sqrt{3}.$$

(b) Hence, find $\dfrac{dB}{dt}$ at this point.

3961. The *Mandelbrot set* is defined using the following iteration, with starting values $x_0 = 0$, $y_0 = 0$, for constants a and b:

$$x_{n+1} = x_n^2 - y_n^2 + a,$$
$$y_{n+1} = 2x_n y_n + b.$$

(a) Setting $b = 0$, find x_2 in terms of a.

(b) Setting $a = 0$, solve for b in the equation

$$x_3 = y_2 + y_3.$$

3962. (a) Show that the equation of the normal to the curve $y = x^3$ at $x = p$ is

$$3p^2y + x - p - 3p^5 = 0.$$

(b) Hence, using the Newton-Raphson method or otherwise, determine which of $(14, 7)$ and $(13, -3)$ is further from the cubic $y = x^3$.

3963. A projectile is launched from ground level, at speed u, at an angle θ above the horizontal.

(a) Find expressions for x and y at time t.

(b) Assuming the projectile travels over flat ground, find its range, in terms of u and θ.

(c) Hence, show that the range of a projectile over flat ground is maximised when $\theta = 45°$.

3964. You are given that $12960x$ and $8640x^2$ are two of the terms of the expansion of $(a + 2x)^b$, where a and b are single-digit natural numbers.

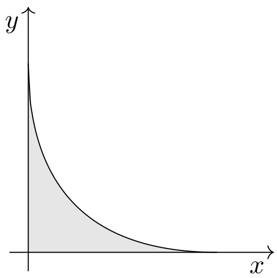
(a) Show that the following equations hold:

$$6480 = ba^{b-1},$$
$$4320 = b(b-1)a^{b-2}.$$

(b) Hence, show that $a = \frac{3}{2}(b-1)$ and that $b - 1$ must therefore be even.

(c) Using the prime factorisation $6480 = 2^4 \cdot 3^4 \cdot 5$, determine the value of a and the value of b.

3965. Determine, in terms of the positive constant c, the area of the region enclosed by the coordinate axes and the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$.



3966. A rational function h is defined by

$$\mathrm{h} : x \longmapsto \frac{16x^4}{2x - 1}.$$

(a) Write down the largest real domain of h.

(b) Express $\mathrm{h}(x)$ as the sum of a polynomial and an algebraic fraction with constant numerator.

(c) The graph $y = \mathrm{h}(x)$ has two stationary points: a maximum at the origin and a minimum at $(2/3, 256/27)$. Determine the range of $\mathrm{h}(x)$.

3967. Prove that $\log_{xy} a \equiv \dfrac{1}{\frac{1}{\log_x a} + \frac{1}{\log_y a}}$.

3968. A particle moves with coordinates, at time $t \geq 0$, given by $x = \sin t$, $y = \sin kt$, for some constant $k \in \mathbb{N}$. Determine whether or not the particle is ever $\sqrt{2}$ units from the origin, in the cases

(a) $k = 2$,

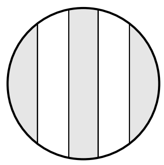
(b) $k = 3$.

3969. Simultaneous equations are defined by

$$R \cot \theta = 2,$$
$$R \operatorname{cosec} \theta = 3.$$

Determine the possible values of R, giving your answers exactly.

3970. A circular icon, of radius 5, is shaded in five stripes of equal width, as shown.



Find, to 3sf, the area of the central stripe.

3971. In a factory, tablets of paracetamol fall vertically from a dispensing chute onto a conveyor belt, which then carries them sideways. The belt moves at 1.4 ms^{-1}, and the tablets emerge from the chute at a steady rate of 2400 grams per second. Let δt be the amount of time taken (assumed constant) for each tablet to accelerate horizontally from rest to 1.4 ms^{-1}.

(a) Find expressions, in terms of δt, for

i. the acceleration of the tablets,

ii. the mass which emerges from the chute over the time interval δt.

(b) Show that the total horizontal force exerted by the belt on the tablets is independent of δt.

3972. A sequence is given, for constants $a, b > 0$, by

$$P_k = |P_{k-1}| - b, \quad P_1 = a.$$

Show that the sequence becomes periodic if

(a) $a < b$,

(b) $b \leq a < 2b$.

3973. Show that the simultaneous equations $y = \operatorname{arccot} x$ and $y = \frac{\pi}{2} - x$ have exactly one (x, y) solution.

3974. Show that the function $\mathrm{f} : x \mapsto x^3 - 4x + \ln x$ is not invertible over the domain $\mathbb{R}^+$.

3975. Sketch the graph $y = (x^2 - 1)^3 - x^2 + 1$.

3976. Sample values, designated $X_1, ..., X_n$, are taken from a large population, which has distribution

$$X \sim N(215, 16^2).$$

Find the smallest value n such that the probability of the sample mean differing from 215 by at least 10 is less than 1%.

3977. A parabola is defined as $y = (x - a)(x - b)$, where $a, b \in \mathbb{R}$. Find the new equation of the parabola when it is

(a) reflected in the line $x = p$,

(b) reflected in the line $y = q$.

3978. This question concerns the definite integral

$$I = \lim_{k \to 0} \int_k^e \ln x \, dx.$$

(a) We can rewrite $\lim_{k \to 0} k \ln k$ as

$$\lim_{k \to 0} \frac{\ln k}{\frac{1}{k}}.$$

Since the numerator and the denominator both diverge, *L'Hôpital's rule* states that the limit can be found by first differentiating numerator and denominator. Show that

$$\lim_{k \to 0} k \ln k = 0.$$

(b) Determine the value of I.

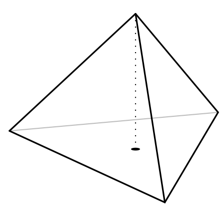
(c) Interpret the value of I graphically.

3979. A particle moves with position at time t given by

$$x = t^2 - t,$$
$$y = t^4.$$

Find the position(s) when the particle is moving in the same direction as the vector $\mathbf{i} + 2\mathbf{j}$.

3980. A regular tetrahedron has four vertices, each at the same distance l from the other three. One of the faces is placed flat against horizontal ground.



Show that the height of the tetrahedron is $\frac{\sqrt{6}}{3} l$.

3981. Three dice are rolled, and the scores are recorded in non-descending order as X, Y, Z. Find

$$\mathbb{P}(Z - X - Y \geq 3).$$

3982. A function has instruction

$$f(x) = \frac{1}{42x^2 - 71x + 30} + 169.$$

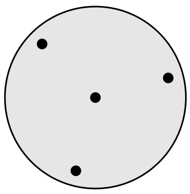
(a) Find the equations of any asymptotes of the graph $y = f(x)$.

(b) Show that $f(x)$ is stationary at $x = \frac{71}{84}$.

(c) Find the range of f over the largest possible real domain.

(d) Explain, with reference to a careful sketch, why the Newton-Raphson method will fail to find a root of the equation $f(x) = 0$ with any integer starting point.

3983. A smooth, uniform, circular lamina of mass m kg is in equilibrium, resting on four supports. The set of four supports is rotationally symmetrical. This is shown below in plan view:



Prove that the forces at the four supports cannot be determined without further information.

3984. Prove that, if $y = f(x)$ has rotational symmetry around the point (a, b), then $y = f'(x)$ has the line $x = a$ as a line of symmetry.

3985. State, with a reason, whether the following hold, regarding the normal distribution $Z \sim N(0, 1)$.

(a) $\mathbb{P}(|Z| > k) > \mathbb{P}(Z > k)$, for all $k \in \mathbb{R}$,

(b) $\mathbb{P}(Z > k + 1) < \mathbb{P}(Z > k)$, for all $k \in \mathbb{R}$.

3986. A function is defined, for some constant k, as

$$f : \begin{cases} x \mapsto k - 3x, & x \in (-\infty, 2) \\ x \mapsto x(x-1)^{-2}, & x \in [2, \infty) \end{cases}$$

(a) Find k such that f has no discontinuity other than at $x = 1$.

(b) Verify that f′ also has no discontinuity other than at $x = 1$.

3987. True or false?

(a) $x^4 - 1 \equiv (x^2 + 1)(x + 1)(x - 1)$,

(b) $x^6 - 1 \equiv (x^3 + 1)(x^2 + 1)(x - 1)$,

(c) $x^8 - 1 \equiv (x^4 + 1)(x^2 + 1)(x^2 - 1)$.

3988. Two inequalities are given as

$$(x - a)(y - b) > 0,$$
$$(x - a)^2 + (y - b)^2 \geq 1.$$

Shade the regions of the (x, y) plane which satisfy both of the inequalities simultaneously.

3989. A wedding car has cans tied behind it, which trail on the road. The strings are 75 cm long, and are tied to the bumper 45 cm above the tarmac. The contact between road and cans is assumed to be perfectly smooth, and air resistance is neglected.

Determine the least acceleration of the car which would cause the cans to leave the tarmac.

3990. The cubic graph $y = 4x^3 - 6x - 7$ is rotated 180° around its point of inflection. Write down the equation of the transformed graph.

3991. Two inequalities are given as follows:

$$xe^y \geq 1,$$
$$x + y \leq 2.$$

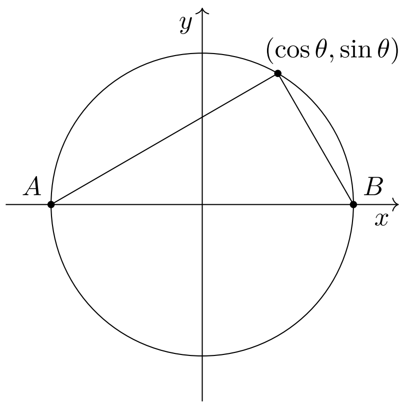
(a) Sketch the boundary equations, and shade the region S whose points satisfy both inequalities simultaneously.

(b) Using a numerical method, find the points of intersection of the boundary equations.

(c) Show that, to 4sf, S has area 1.949.

3992. Show that the value of $f(x) = x^4 - x^2 + 2x + 1$ is always at least twice the value of its input.

3993. This question concerns a proof of the angle in a semicircle theorem. Let P be $(\cos\theta, \sin\theta)$, and A and B be $(-1, 0)$ and $(1, 0)$:



(a) Show that the gradient of AP is $\dfrac{\sin\theta}{\cos\theta + 1}$.

(b) Find a similar expression for BP.

(c) Hence, prove that $\angle APB = 90°$.

3994. Solve the differential equation $y' \cot x = y$, given the initial conditions $x = 0, y = 2$.

3995. In this question, $x, y \in \mathbb{N}$. Prove by contradiction that, if there exist $a, b \in \mathbb{N}$ such that $ax + by = 1$, then x and y have no common factor other than 1.

3996. In *Bernoulli's equation*, the function g satisfies

$$\mathrm{g}'(x) = \mathrm{g}(x) + \mathrm{g}(x)^2.$$

You are given that $\mathrm{g}(0) = 1$.

(a) Show that $\mathrm{g}(x) = \dfrac{e^x}{2 - e^x}$.

(b) Show that this function is increasing.

3997. In this question, vectors $\mathbf{a}, \mathbf{b}$ are perpendicular. Two particles have position at time t given by

$$\mathbf{r}_1 = t^2\mathbf{a} + (2 - t)\mathbf{b},$$
$$\mathbf{r}_2 = (2t - 1)\mathbf{a} + t^2\mathbf{b}.$$

Show that the particles collide.

3998. The tangent to $y = x^{\frac{1}{3}}$ at $x = a$ crosses the x axis at $-2/5$. By finding the equation of a general tangent line, determine a.

3999. Prove the following result, for $|r| < 1$,

$$\frac{d}{dt}\left(\sum_{i=1}^{\infty} ar^{i-1}\right) = \frac{(1-r)\frac{da}{dt} + a\frac{dr}{dt}}{(1-r)^2}.$$

4000. A curve C is defined by $y = \sqrt{x^4 + 1}$.

(a) Show that C has a local minimum at $x = 0$.

(b) Show that, for large x, C approaches $y = x^2$.

(c) Hence, sketch C.

——— End of Volume IV ———